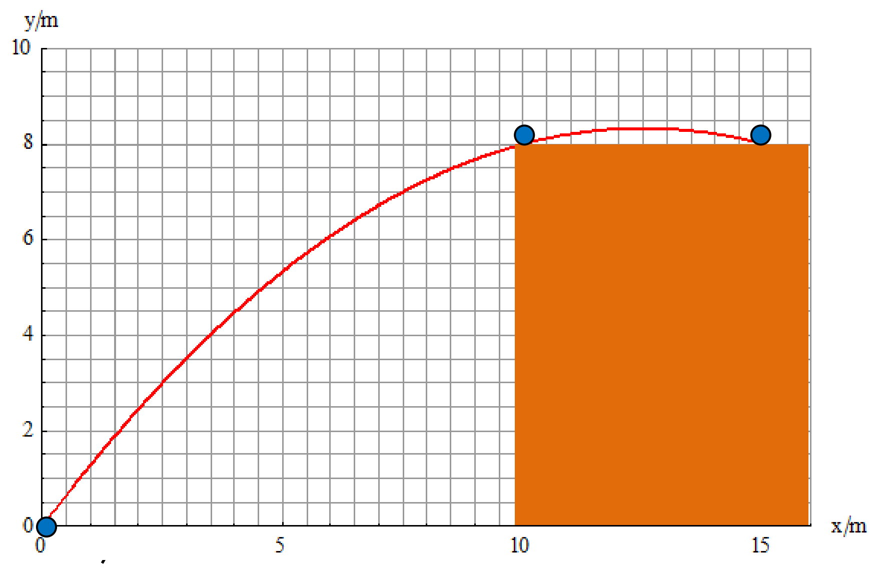
Teacher notes

Topic A

Walter Lewin's problem: an interesting projectile motion problem

A ball is to be launched so that it lands on a building that is 8 m high. The building's left wall is 10 m from the launch point. The ball is required to land 15 m away from the launch point. What is the **least** time taken for the ball to land and what is the launch speed and angle?



For different combinations of initial speed and angle, there are many paths that will take the ball from (0, 0) to (15, 8).

One of them is shown above: it just misses the wall. **If we assume** that this is the path that will give us the least travel time then:

The parabolic path must have equation $y = ax^2 + bx$. Since it must pass through (10, 8) and (15, 8) we easily find $a = -\frac{4}{75}$ and $b = \frac{4}{3}$ so that the path equation is $y = -\frac{4}{75}x^2 + \frac{4}{3}x$. Now, in general we have $x = ut\cos\theta$ and $y = ut\sin\theta - \frac{1}{2}gt^2$. Solving the first equation for time we find $t = \frac{x}{u\cos\theta}$ and substituting in the second we get the equation of the path:

$$y = u(\frac{x}{u\cos\theta})\sin\theta - \frac{1}{2}g(\frac{x}{u\cos\theta})^2$$
$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Comparing with $y = -\frac{4}{75}x^2 + \frac{4}{3}x$ gives

$\tan\theta = \frac{4}{3}$ and $u = \frac{25\sqrt{15}}{6} \approx 16.137\,\text{ms}^{-1}$. From $\cos\theta = \frac{15}{vt}$, $t = 1.549 \approx 1.5\,\text{s}$.

But why is this, the path of least time? If you want you may read on!

$$x = 15 = ut\cos\theta$$
$$y = 8 = ut\sin\theta - 5t^2$$

From the first equation $\cos\theta = \frac{15}{ut}$. The second equation gives

$\left(8 + 5t^2\right)^2 = u^2t^2\sin^2\theta = u^2t^2(1 - \frac{225}{u^2t^2})$ i.e.

$64 + 80t^2 + 25t^4 = u^2t^2 - 225$ or $25t^4 - (v^2 - 80)t^2 + 289 = 0$. The solutions are

$$t^2 = \frac{u^2 - 80 \pm \sqrt{(u^2-80)^2 - 100\times 289}}{50}.$$

We seek the least possible value of time so we take the minus sign:

$$t^2 = \frac{u^2 - 80 - \sqrt{(u^2-80)^2 - 100\times 289}}{50} \qquad \text{Eqn (1)}$$

But we must ensure that we don't hit the wall: the path equation is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}.$$

Substituting $x = 10, y \geq 8$ and using $\cos\theta = \frac{15}{ut}$ **where t is the time to reach x = 15** gives

$$10\tan\theta - \frac{500}{u^2\cos^2\theta} \geq 8$$

$$10\frac{\sqrt{u^2t^2-225}}{15} - \frac{500u^2t^2}{225} \geq 8$$

$$10\frac{\sqrt{u^2t^2-225}}{15} \geq 8 + \frac{500u^2t^2}{225}$$

$$\left(\frac{2}{3}\sqrt{u^2t^2-225}\right)^2 \geq \left(8+\frac{500u^2t^2}{225}\right)^2$$

$$(\frac{500}{225})^2 t^4 - (\frac{4u^2-320}{9})t^2 + 164 \leq 0$$

Hence $(\frac{500}{225})^2 t^4 - (\frac{4u^2-320}{9})t^2 + 164 \leq 0$ and so:

$$\frac{\frac{4u^2-320}{9} - \sqrt{(\frac{4u^2-320}{9})^2 - 4\times(\frac{500}{225})^2\times 164}}{2\times(\frac{500}{225})^2} \leq t^2 \leq \frac{\frac{4u^2-320}{9} + \sqrt{(\frac{4u^2-320}{9})^2 - 4\times(\frac{500}{225})^2\times 164}}{2\times(\frac{500}{225})^2}$$

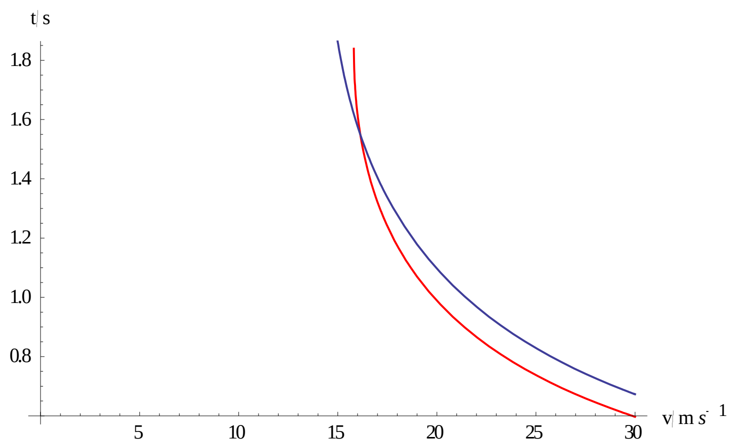
Again we seek the least value of time so

$$t^2 = \frac{\frac{4u^2-320}{9} - \sqrt{(\frac{4v^2-320}{9})^2 - 4\times(\frac{500}{225})^2\times 164}}{2\times(\frac{500}{225})^2} \quad \text{(Eqn 2)}$$

This proves that the least time is obtained when the projectile just misses the wall at *x* =10, *y* = 8.

The least time to reach the target must satisfy **both equations (1) and (2)**.

Graphing both as functions of the launch speed *v* gives the following graphs, where blue is eqn 1 and red is eqn 2:



The two curves intersect at the solution which is t = 1.549 s and $u = 16.137\,\mathrm{ms}^{-1}$. This implies

$$\cos\theta = \frac{15}{ut} = \frac{15}{16.137 \times 1.549} = \frac{3}{5} \Rightarrow \theta = 53.13^{\circ}.$$